Comment on 'A critical understanding of the fractal model of metabolic scaling'

Chaui-Berlinck recently published a paper in which he claims that the original West, Brown and Enquist (WBE) model for metabolic scaling (West et al., 1997) is fundamentally flawed (Chaui-Berlinck, 2006). In particular, Chaui-Berlinck asserted that '*the minimization procedure* [of the original WBE model] *is mathematically incorrect and ill-posed*' and that the model '*lacks self-consistency and correct statement*'. These are strong accusations and should, therefore, be closely scrutinized. Unfortunately, Chaui-Berlinck's conclusions are incorrect because of rudimentary mathematical mistakes, and, even worse, these false conclusions are now being perpetuated in the literature. For example, Muller-Landau (Muller-Landau, 2007), in a review for Faculty of 1000, recently drew attention to Chaui-Berlinck's paper by stating that '*This article carefully dissects West, Brown and Enquist's (1997) derivation of allometric scaling of metabolism. It illuminates important logical inconsistencies and mathematical problems with the argument*'.

We note that none of the original authors nor the extended scaling community associated with the WBE model were asked to review Chaui-Berlinck's manuscript. As we show below, the entire basis of Chaui-Berlinck's paper stems from fundamental mathematical mistakes. In short, the conclusions of Chaui-Berlinck (and, subsequently, Muller-Landau) are completely incorrect. We conclude that Chaui-Berlinck's paper (Chaui-Berlinck, 2006) should be retracted.

The most egregious errors of Chaui-Berlinck are seen in his equation 5a. Specifically, Chaui-Berlinck makes two mistakes. He first mis-transcribes the original equation from WBE (West et al., 1997) and then makes a fundamental error in his calculus.

In his equation 5a, Chaui-Berlinck insists that he is carefully analyzing the mathematics of the WBE model. He obtains the quotient 0/0 in several equations and then concludes that, because of his analysis, the results of WBE are meaningless. However, Chaui-Berlinck's results of 0/0 only demonstrate both a misreading of the WBE paper and a basic mathematical error. The first mistake stems from Chaui-Berlinck incorrectly writing equation 9 from WBE as:

$$V_\mathrm{b} = \frac{V_\mathrm{c}}{(\beta_>^2\gamma)^N}\left\{\left(\frac{\beta_>}{\beta_<}\right)^{2\bar{k}}\left[\frac{1-(n\beta_>^2\gamma)^{\bar{k}}}{1-(n\beta_>^2\gamma)}\right]+\left[\frac{1-(n\beta_>^2\gamma)^{N}}{1-(n\beta_>^2\gamma)}\right]-\left[\frac{1-(n\beta_>^2\gamma)^{\bar{k}}}{1-(n\beta_>^2\gamma)}\right]\right\}, \qquad (1a)$$

but the correct expression, as given in WBE, is:

$$V_\mathrm{b} = \frac{V_\mathrm{c}}{(\beta_>^2\gamma)^N}\left\{\left(\frac{\beta_>}{\beta_<}\right)^{2\bar{k}}\left[\frac{1-(n\beta_<^2\gamma)^{\bar{k}}}{1-(n\beta_<^2\gamma)}\right]+\left[\frac{1-(n\beta_>^2\gamma)^{N}}{1-(n\beta_>^2\gamma)}\right]-\left[\frac{1-(n\beta_>^2\gamma)^{\bar{k}}}{1-(n\beta_>^2\gamma)}\right]\right\}. \qquad (1b)$$

Notice that he swaps two $\beta_<$ for $\beta_>$ in the first term inside the parentheses. For the second mistake, he then goes on to evaluate equation 9 from WBE *in a regime where the equation does not hold*. Thus, Eqn 1b (above) contains expressions for geometric sums that hold only for values of $\beta_<$, $\beta_>$, n and γ such that $n\beta_<^2\gamma\neq1$ and $n\beta_>^2\gamma\neq1$. These expressions are not stated explicitly in WBE because they are apparent from basic rules for sums. The correct result can be obtained directly from equation 9 in the original WBE paper (or Eqn 1b here) by taking the limit $\beta_>\rightarrow n^{-1/3}$, corresponding to $n\beta_>^2\gamma\rightarrow1$ (because $\gamma=n^{-1/3}$). Although it is true that the numerator and denominator of the second two terms inside the parentheses both go to zero in this limit, this does *not* equal the limit of the fraction. From introductory calculus, the limit of the fraction as a whole can be obtained using L'hospital's rule (for example, see http://mathworld.wolfram.com/LHospitalsRule.html or even a standard calculus class website such as http://www.math.tamu.edu/~fulling/coalweb/lhop.htm). Using L'hospital's rule simply amounts to taking the derivative of the numerator and denominator separately and *only then* taking the limit of the numerator and denominator in the resultant fraction.

As was done in the original WBE model, a finite geometric sum can be expressed as:

$$\sum_{k=0}^{k=N-1} x^k = \frac{1-x^N}{1-x}\,,\ x\neq1\,. \qquad (2)$$

The correct result for x=1 is found by taking the limit $x\rightarrow1$ using L'hospital's rule:

$$\lim_{x\to1}\frac{1-x^N}{1-x} = \frac{\lim\limits_{x\to1}\left[\dfrac{d}{dx}\,(1-x^N)\right]}{\lim\limits_{x\to1}\left[\dfrac{d}{dx}\,(1-x)\right]} = \frac{\lim\limits_{x\to1}(-Nx^{N-1})}{\lim\limits_{x\to1}(-1)} = N\,, \qquad (3)$$

or going back to the original sum and recognizing that:

$$\sum_{k=0}^{N-1} 1^k = \sum_{k=0}^{N-1} 1 = N\,. \qquad (4)$$

Unfortunately, Chaui-Berlinck's criticism did not incorporate these rules.

Now realizing that we can think of x as $n\beta_>^2\gamma$, the sum from 0 to N–1 as over the N levels of the branching network, and using $\beta_<=n^{-1/2}$ along with our previous expressions for the other scaling ratios, equation 9 from WBE (i.e. Eqn 1b here) gives the correct result:

$$V_\mathrm{b} = V_\mathrm{c}n^{4N/3}\left[\frac{n^{-(N-\bar{k})/3}-n^{-N/3}}{1-n^{-1/3}}+n^{-N/3}(N-\bar{k})\right] = V_\mathrm{c}n^{4N/3}\left(\frac{n^{-\bar{N}/3}-n^{-N/3}}{1-n^{-1/3}}+n^{-N/3}\,\bar{N}\right). \qquad (5)$$

Here, $\bar{N}=N-\bar{k}$, as originally reported in WBE. When $N \gg \bar{N}$ and $\bar{k} \gg 1$, with V_c and $\bar{N}$ constant, we have the original WBE prediction, $M \propto V_b \propto n^{4N/3} \propto N_c^{4/3} \propto B^{4/3}$, where N_c is the number of capillaries in the organism and is directly related to metabolic rate. Consequently, the most critical claim made by Chaui-Berlinck is patently false.

Chaui-Berlinck makes several additional errors. In the equation at the top of the second column on p. 3050, Chaui-Berlinck's treatment of the geometric constants in the Lagrange multiplier calculation is not correct. Specifically, in the original WBE model, $(4/3)\pi(l/2)^3$ is the service volume, and the geometric constant $(4/3)\pi(1/2)^3$ is absorbed into the arbitrary constant λ_k, highlighting the fact that the distinction between a sphere and a cube does not matter for these arguments. Lastly, Chaui-Berlinck rehashes the mistaken ideas of Dodds et al. (Dodds et al., 2001) and Kozlowski and Konarzewski (Kozlowski and Konarzewski, 2004). Interestingly, Chaui-Berlinck perpetuates these flawed arguments once more but does not cite the responses, which does not present a balanced, fair or accurate view of the field (Brown et al., 2005; Savage et al., 2004). In summary, Chaui-Berlinck's paper is riddled with mathematical mistakes that reflect a misreading of the original WBE paper.

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