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A THEORETICAL TREATMENT OF THE REFLEXION OF LIGHT BY MULTILAYER STRUCTURES

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1. INTRODUCTION

Interference between rays of light reflected at successive interfaces in a laminated structure has often been suggested as one of the methods by which colours of animals might be produced (e.g. Biedermann, 1914, p. 1892; Rayleigh, 1919; Onslow, 1921, p. 13; Fox & Vevers, 1960). An even more striking phenomenon is the very high reflectivity, over a substantial part of the visible spectrum, found, for example, in fish scales (Denton & Nicol, 1966), in the argentea of the eye of the scallop, *Pecten* (Dakin, 1910; Land, 1965) and in tapeta of the eyes of cartilaginous fishes (Denton & Nicol, 1964). In several of these cases it has recently been shown (Land (1966), for the eye of *Pecten*; Denton & Land (1967), for fish scales) that the reflecting structure consists of a stack of flat transparent crystals, each having an optical thickness close to a quarter of a wavelength and separated from its neighbours by layers of cytoplasm of equal optical thickness. Constructive interference therefore occurs between light reflected at successive interfaces, and a high reflectivity is obtained in the same way as in artificial 'dielectric reflectors' made by vacuum deposition of layers of transparent materials of alternately high and low refractive index. Other highly reflecting structures, such as the tapeta behind the retinae of some mammals, also have regularly repeating structural elements at spacings of the right order of magnitude for the same kind of interference to occur, but have not yet been worked out from the optical point of view (e.g. Pedler, 1963; Dartnall *et al.* 1965).

It might have been thought that the commercial development of multilayer dielectric reflectors and interference filters would have led to the publication of convenient theoretical expressions for dealing with biological systems of this kind. Discussions with Dr M. F. Land during his work on the argentea of the eye of *Pecten* showed however that this was apparently not the case, and the author derived a number of simple formulae, relevant to reflectors of this kind, which are not given in the accounts of multilayer theory by, for instance, Heavens (1955, 1960), Vašiček (1960), Born & Wolf (1964) or Baumeister (1965). These formulae can presumably be obtained by the powerful matrix method of Abelès (1950), but this method is of little use to those who are not familiar with matrix algebra; another advantage of the method used in the present paper is that it gives greater insight into the processes which give rise to the optical properties of the complete stack of plates than does the method of Abelès. For these reasons it seemed likely that the method, and the formulae obtained by means of it, would be useful to biologists investigating other multilayer reflecting structures.

Most of the steps in this method are the same as were used by Rayleigh (1917); he in turn was adapting for thin layers the method by which Stokes (1862) had treated the case in which the plates are thick enough for interference phenomena to be disregarded. These papers seem to have been to a large extent forgotten; of the authors quoted in the preceding paragraph, only Abelès (1950) refers to them.

Rayleigh did not, however, derive all the particular formulae which will be given in the present paper. The author felt it would also be useful to present the whole method afresh in a more elementary style than Rayleigh's treatment.

In § 2, the problem is stated in its simplest form (optical thickness of spaces equal to that of plates; normal incidence; no reflexion from different materials above or below the stack itself), and symbols are defined. In § 3, the steps of the solution of this problem are outlined and the most important equations are given without proof; these equations and others are derived in § 4. In § 5 the results are extended to some more general cases.

Section 4.3 contains a discussion of the variation of phase and amplitude through the thickness of the stack.

It is assumed throughout that absorption of light is negligible. The results could no doubt be adapted to the case of finite absorption by using complex values for the phase retardations due to the plates and the spaces between them. The method is restricted to stacks consisting of a regularly repeated sequence of layers.

2. STATEMENT OF PROBLEM, AND DEFINITIONS

2.1. *The problem*

Figure 1 illustrates the situation that is dealt with in §§ 3 and 4. Light is incident normally from above on a stack of transparent plates, p in number, uniformly spaced in an infinite medium of different refractive index. The problem is to find the fraction of the incident intensity that is reflected.

The plates are all of equal thickness d_b, and their refractive index is n_b, so that the phase retardation due to traversing a single plate once is given by

$$\phi_b = (2\pi/\lambda)\, d_b n_b,$$

where λ is the wavelength of the light *in vacuo*.

The spaces have thickness d_a and refractive index n_a, so the phase retardation due to traversing a single space is

$$\phi_a = (2\pi/\lambda)\, d_a n_a.$$

The a's, denoting the amplitudes (and phases) of the light at different levels within the stack are defined at the broken lines in Fig. 1, i.e. at the centres of the spaces between the plates and at corresponding levels above the top plate and below the bottom plate. This convention is chosen because it makes the structure between any two of the broken lines symmetrical, and therefore simplifies the equations.

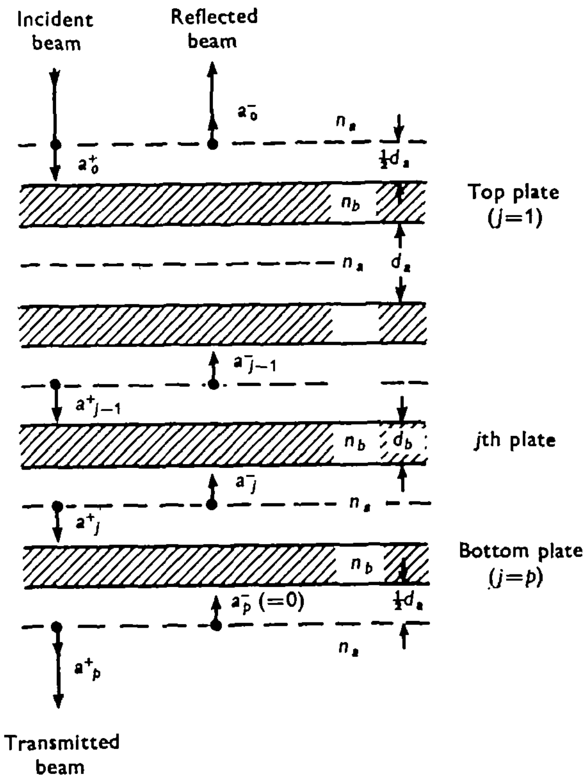

Fig. 1. Diagram to illustrate the situation treated in §§ 3 and 4. Light is incident normally from above on a stack of plates, p in number, with refractive index n_b, immersed in a medium of refractive index n_a. a_j^+ denotes amplitude of component propagating in the direction of the incident beam, and a_j^-, amplitude of component in the opposite direction, both measured at the centre of the space below the jth plate. Drawn for $p = 4$, $j = 3$.

2.2. *Definitions*

Only the symbols which are used throughout the paper are defined here; a number of other symbols are defined where they arise in §§ 4.1, 5.2 and 5.3.

Amplitudes

See Fig. 1. Defined as amplitudes of electric vector, as a complex quantity so as to indicate phase as well as absolute amplitude.

a_j^+ : downward wave, at centre of space below jth plate.
a_j^- : upward wave, at centre of space below jth plate.
a_i, a_r, a_t: defined in § 4.1 and Fig. 2.

Refractive index (n) *and thickness* (d) *of layers*

n_b, d_b: each plate in the stack.
n_a, d_a: each space between adjacent plates.
n_0, d_0; n_z, d_p: defined in § 5.3 and Fig. 4.

Phase retardations for light of wavelength λ

Expressions are for normal incidence; oblique incidence is dealt with in § 5.2.

ϕ_b: in each plate of the stack, $= 2\pi d_b n_b/\lambda$.

ϕ_a: in each space between plates, $= 2\pi d_a n_a/\lambda$.

ϕ: equal to ϕ_a or ϕ_b when $\phi_a = \phi_b$.

δ: defined in § 4.1.

ϕ_0, ϕ_s, χ: defined in § 5.3.

Amplitude reflexion (r) *and transmission* (t) *coefficients*

$r_1, t_1, r_2, t_2, r_3, t_3, R, T$ are defined for light incident from above; the same symbols with primes are for light incident from below on the same interfaces or structures. In general, complex, but r, t, r_0, r_s are real.

r, t: at n_b/n_a interface.

r_0: at n_0/n_a interface (§ 5.3).

r_s: at n_a/n_s interface (§ 5.3).

ρ, τ: one plate of the stack, with one half-space above and below.

R, T: whole stack, with one half-space above the top plate and one half-space below the bottom plate.

$r_1, r_2, r_3, t_1, t_2, t_3$: defined in § 4.1 (see also Fig. 2).

R_s, R_{0s}: defined in § 5.3.

Other symbols

μ, μ_1, μ_2: ratio of amplitudes in successive layers, defined by equations (6)–(10), (15), (20), (21), (23), (24), (44). ($|\mu_1| < |\mu_2|$ when μ's are real.)

h, h_1, h_2: ratio of upgoing to downgoing wave, defined by equations (7)–(9), (11), (12), (16), (17), (22), (26), (45), (46). ($|h_1| < |h_2|$ when h's are real.)

m: $= \mu_1^p$, and equation (32).

α: defined by equations (13) and (14).

θ: defined by equations (25) or (49).

$2k$: coefficient of h in equation (12), (17) or (46).

$2k'$: coefficient of μ in equations (10), (15) or (44).

p: number of plates in the stack.

j: ordinal number of a plate in the stack, from $j = 1$ for top plate to $j = p$ for bottom plate.

ν_a, ν_b: defined in § 5.2.

The formulae developed in this paper were used by Land (1966) in connexion with the reflecting layer in the eye of *Pecten*. The symbols used by him differ from those used in the present paper in the respects shown in Table 1.

3. OUTLINE OF THE METHOD

The main steps in the procedure are listed here, numbered to agree with the subsections of § 4 in which the formulae are derived.

The expressions given in this section are appropriate to the case where the optical thicknesses of plates and spaces are equal, incidence is normal to the interfaces, and there is no reflexion from structures above the topmost plate or below the bottom one. Extensions of these results to more general cases are given in § 5.

The equations are numbered according to their sequence in § 4.

Table 1

Symbol used in Land (1966)	Equivalent in this paper
n_1	n_b
n_2	n_a
t_1	d_b
t_2	d_a
k	$2p$
r	r^2
R	$\|R\|^2$
y	m^2 or $1/m^2$ according to the sign of $\sin\phi$

Step 1

Obtain expressions for the reflexion and transmission coefficients of a single plate, taking account of multiple reflexions within the plate. They are respectively:

$$\rho = -r \exp(-i\phi) \frac{1-\exp(-2i\phi)}{1-r^2 \exp(-2i\phi)} \tag{1c}$$

and

$$\tau = \frac{(1-r^2)\exp(-2i\phi)}{1-r^2\exp(-2i\phi)}. \tag{2c}$$

Step 2

Derive a pair of recurrence relations connecting the amplitudes of the upward- and downward-propagating waves at successive layers in the stack.

They are
$$a_{j-1}^- = \rho a_{j-1}^+ + \tau a_j^- \tag{4}$$

and
$$a_j^+ = \tau a_{j-1}^+ + \rho a_j^-. \tag{5}$$

Step 3

Show that these equations are solved by

$$a_j^+ = \alpha a_0^+ \mu_1^j + (1-\alpha) a_0^+ \mu_2^j, \tag{13}$$

and
$$a_j^- = \alpha h_1 a_0^+ \mu_1^j + (1-\alpha) h_2 a_0^+ \mu_2^j, \tag{14}$$

where

(*a*) μ_1 and μ_2 are the solutions of the equation

$$\mu^2 - \frac{1+\tau^2-\rho^2}{\tau}\mu + 1 = 0, \tag{10}$$

(*b*) h_1 and h_2 are obtained by putting $\mu = \mu_1$ or $\mu = \mu_2$ in the equation

$$h = \frac{\tau\mu - (\tau^2-\rho^2)}{\rho}, \tag{11}$$

and (*c*) α is a constant whose value depends on the boundary conditions.

Step 4

Obtain α by considering the boundary condition at the bottom of the stack. The result is

$$\alpha = \frac{h_2}{h_2 - m^2 h_1}, \tag{27}$$

where

$$m = \mu_1^p. \tag{28}$$

Step 5

The amplitude reflexion coefficient $R\,(= a_0^-/a_0^+)$ is now obtained from equation (14). It is

$$R = \frac{1 - m^2}{h_2 - m^2 h_1}. \tag{29}$$

This is a complex quantity, indicating the phase as well as the absolute amplitude of the reflected light.

Step 6

Take the square of the modulus of R to obtain the reflectance of the stack (fraction of incident intensity that is reflected).

When the optical thickness of each plate (and each space) is close enough to a quarter wavelength so that $\cos^2\phi < r^2$, then m is real and

$$|R|^2 = 1 \Big/ 1 + \frac{4m^2(r^2 - \cos^2\phi)}{r^2(1 - m^2)^2}. \tag{35}$$

This approaches unity (complete reflexion) as the number of plates is increased.

When the optical thicknesses deviate from a quarter wavelength far enough so that $\cos^2\phi > r^2$, then m and the μ's are complex, and

$$|R|^2 = 1 \Big/ 1 + \frac{\cos^2\phi - r^2}{r^2 \sin^2 p\theta}, \tag{39}$$

where

$$\cos\theta = \frac{\cos 2\phi - r^2}{1 - r^2}. \tag{25}$$

The right-hand side of equation (39) is an oscillatory function of θ, which in turn varies with ϕ and therefore with wavelength. When the number of plates p is large, it is convenient to average this function over one cycle of its oscillation; the result is

$$\overline{|R|^2} = 1 - \sqrt{\left(1 - \frac{r^2}{\cos^2\phi}\right)}. \tag{40}$$

When $\cos^2\phi = r^2$, equations (35) and (39) reduce to

$$|R|^2 = 1 \Big/ 1 + \frac{1 - r^2}{4p^2 r^2}. \tag{43}$$

4. DERIVATION OF EQUATIONS

4.1. *Reflexion by a single plate*

Suppose that two partially reflecting surfaces are separated by a layer of material which introduces a phase delay of δ (Fig. 2).

r_1, t_1, r_2 and t_2 are the reflexion and transmission coefficients for light incident from above, and r'_1, t'_1, r'_2 and t'_2 are the corresponding quantities for light incident from below.

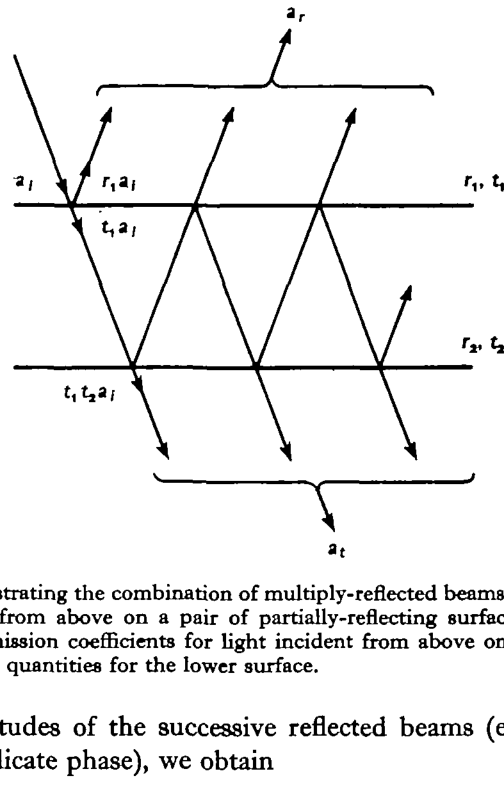

Fig. 2. Diagram illustrating the combination of multiply-reflected beams when light of amplitude a_i is incident from above on a pair of partially-reflecting surfaces. r_1, t_1: amplitude reflexion and transmission coefficients for light incident from above on the upper surface; r_2, t_2: corresponding quantities for the lower surface.

Adding the amplitudes of the successive reflected beams (expressed as complex numbers so as to indicate phase), we obtain

$$a_r = a_i\{r_1 + t_1 r_2 t'_1 \exp(-2i\delta) + t_1 r_2 r'_1 r_2 t'_1 \exp(-4i\delta) + \ldots\}.$$

The second and later terms between the curly brackets form a geometrical progression which can be summed to give

$$a_r = a_i\left\{r_1 + \frac{r_2 t_1 t'_1 \exp(-2i\delta)}{1 - r'_1 r_2 \exp(-2i\delta)}\right\}.$$

Now a_r/a_i is the combined reflexion coefficient r_3, so

$$r_3 = \frac{r_1 + r_2(t_1 t'_1 - r_1 r'_1)\exp(-2i\delta)}{1 - r'_1 r_2 \exp(-2i\delta)}. \qquad (1)$$

In the same way

$$a_t = a_i t_1 t_2 \exp(-i\delta)\{1 + r_2 r'_1 \exp(-2i\delta) + (r_2 r'_1 \exp(-2i\delta))^2 + \ldots\},$$

or

$$t_3 = \frac{a_t}{a_i} = \frac{t_1 t_2 \exp(-i\delta)}{1 - r'_1 r_2 \exp(-2i\delta)}. \qquad (2)$$

When the two reflectors are simple interfaces between media of different refractive index, the r's and t's are connected by

$$r'_1 = -r_1; \quad r'_2 = -r_2; \quad t_1 t'_1 = 1-r_1^2; \quad t_2 t'_2 = 1-r_2^2$$

(e.g. Born & Wolf (1964), p. 324), so that

$$r_3 = \frac{r_1+r_2 \exp(-2i\delta)}{1+r_1 r_2 \exp(-2i\delta)} \tag{1a}$$

and

$$t_3 = \frac{t_1 t_2 \exp(-i\delta)}{1+r_1 r_2 \exp(-2i\delta)}. \tag{2a}$$

These well-known results give the phases of the reflected and transmitted beams as they just leave the plate, relative to the phase of the incident beam as it just reaches the plate. To apply them to the amplitudes in our stack, we note that both for a_{j-1}^- and for a_j^+ (see Fig. 1) relative to a_{j-1}^+, there are two half-spaces to be traversed in addition to the single plate, giving an extra phase delay of ϕ_a. If r is the reflexion coefficient at an n_b/n_a interface, then by Young's formula

$$r = \frac{n_b-n_a}{n_b+n_a}, \tag{3}$$

and

$$r_1 = -r; \quad r_2 = +r; \quad t_1 = t'_2,$$

so that $t_1 t_2 = 1-r^2$. Defining ρ as the reflexion coefficient for one plate flanked by two half-spaces, and τ as the corresponding transmission coefficient, equations (1 a) and (2 a) become:

$$\rho = \frac{-r \exp(-i\phi_a)\,[1- \exp(-2i\phi_b)]}{1-r^2 \exp(-2i\phi_b)} \tag{1b}$$

and

$$\tau = \frac{(1-r^2) \exp(-i(\phi_a+\phi_b))}{1-r^2 \exp(-2i\phi_b)}. \tag{2b}$$

Since the system now under consideration (one plate flanked by two half-spaces) is symmetrical about the centre of the plate, these expressions for ρ and τ are appropriate for light incident either from above or from below (i.e. $\rho' = \rho$ and $\tau' = \tau$).

In the rest of § 4 only the case $\phi_a = \phi_b$ will be considered (equal optical thicknesses in plates and spaces). Writing $\phi = \phi_a = \phi_b$, equations (1 b) and (2 b) become

$$\rho = \frac{-r \exp(-i\phi)\,[1- \exp(-2i\phi)]}{1-r^2 \exp(-2i\phi)} \tag{1c}$$

and

$$\tau = \frac{(1-r^2) \exp(-2i\phi)}{1-r^2 \exp(-2i\phi)}. \tag{2c}$$

4.2. *The recurrence relations*

When this same plate, the jth, is in place in the stack, light is incident on it from below as well as from above because of reflexion from lower layers in the stack (Fig. 3).

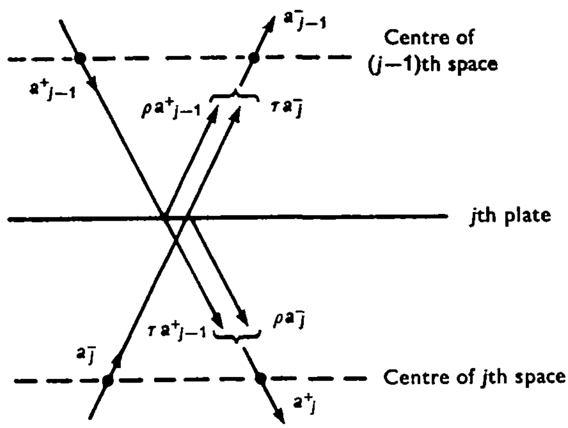

Fig. 3. Derivation of recurrence relations between amplitudes at successive levels in the stack. The *j*th plate is indicated as a single line; the effect of its thickness is incorporated in the expressions for ρ and τ (equations (1*b*) and (2*b*), or (1*c*) and (2*c*)).

The amplitude a_{j-1}^- of light leaving it in the upward direction is the resultant of a component from a_{j-1}^+ reflected by the plate and a component from a_j^- transmitted by the plate, or

$$a_{j-1}^- = \rho a_{j-1}^+ + \tau a_j^-. \quad (4)$$

Similarly, for the downward wave leaving the plate,

$$a_j^+ = \tau a_{j-1}^+ + \rho a_j^-. \quad (5)$$

These equations are the required recurrence relations.

4.3. *Solution of the recurrence equations*

Try whether equations (4) and (5) are solved by equations of the form:

$$a_j^+ = a_0^+ \mu^j \quad (6)$$

and

$$a_j^- = a_0^- \mu^j = h a_0^+ \mu^j. \quad (7)$$

Insert these expressions into 4 and 5 and divide through by $a_0^+ \mu^{j-1}$, obtaining:

$$h = \rho + \tau\mu h \quad (8)$$

and

$$\mu = \tau + \rho\mu h. \quad (9)$$

Eliminating h between these two equations gives the following quadratic for μ:

$$\mu^2 - \frac{1+\tau^2-\rho^2}{\tau}\mu + 1 = 0. \quad (10)$$

Equations (4) and (5) are therefore solved by equations (6) and (7), provided that μ is put equal to either μ_1 or μ_2, the two solutions of equation (10). The corresponding values of h are obtained by substituting $\mu = \mu_1$ or $\mu = \mu_2$ into the following equation:

$$h = \frac{\tau\mu - (\tau^2-\rho^2)}{\rho}, \quad (11)$$

which is obtained by eliminating the μh terms between equations (8) and (9).

Also, μ can be eliminated between equations (8) and (9) to obtain the following quadratic for h:

$$h^2 - \frac{1-(\tau^2-\rho^2)}{\rho} h + 1 = 0. \tag{12}$$

Equations (6) and (7) are therefore solutions of equations (4) and (5) provided that either

$$\mu = \mu_1 \quad \text{and} \quad h = h_1,$$

or

$$\mu = \mu_2 \quad \text{and} \quad h = h_2.$$

Since equations (4) and (5) are linear, any constant multiple of a solution, or any sum of solutions, is also a solution, and a general solution is therefore:

$$a_j^+ = \alpha a_0^+ \mu_1^j + (1-\alpha) a_0^+ \mu_2^j \tag{13}$$

and

$$a_j^- = \alpha h_1 a_0^+ \mu_1^j + (1-\alpha) h_2 a_0^+ \mu_2^j. \tag{14}$$

Equations (10) to (12) give the coefficients in terms of ρ and τ; they may be re-written as functions of r and ϕ by substituting from equations (1c) and (2c), giving respectively:

$$\mu^2 - 2\frac{\cos 2\phi - r^2}{1-r^2}\mu + 1 = 0, \tag{15}$$

$$h = -\frac{\cos\phi}{r} + \frac{i\{\mu(1-r^2)-(\cos 2\phi - r^2)\}}{2r\sin\phi} \tag{16}$$

and

$$h^2 + \frac{2\cos\phi}{r} h + 1 = 0. \tag{17}$$

The condition for real roots in equation (15) is that

$$\left(\frac{\cos 2\phi - r^2}{1-r^2}\right)^2 > 1;$$

this reduces to

$$\cos^2\phi < r^2, \tag{18}$$

which is also the condition that the roots of equation (17) (for h) should be complex.

Hence, whenever the optical thickness of each plate and each space is close enough to a quarter wavelength (or to any odd number of quarter wavelengths) so that $\cos^2\phi < r^2$, then the μ's are real and the h's are complex. Further, since the coefficients of h^2 and h^0 in equation (17) are equal,

$$h_1 h_2 = 1 \tag{19}$$

always, and when the h's are complex, $|h_1| = |h_2| = 1$. Thus the upward wave has the same amplitude as the downward one that it accompanies, but it is shifted in phase. The amplitudes vary exponentially with distance through the stack; in an infinite stack, only the solution with $|\mu| < 1$ can exist and its amplitude decays away toward zero, all the energy being reflected. The solution with $|\mu| > 1$ exists only in so far as energy is reflected at the bottom of the stack. Since μ is real, the phase of each wave is the same or 180° out at every layer in the stack; this means that the velocity of the wave is being pulled by the periodic structure so that an integral number of half-waves corresponds exactly to one repeat, even when ϕ is not exactly $\frac{1}{2}\pi$.

If the wavelength is changed further from the ideal value which makes ϕ exactly $\frac{1}{2}\pi$ (or an odd multiple of $\frac{1}{2}\pi$), $\cos\phi$ increases in either the positive or negative direction until $\cos^2\phi > r^2$. Now the h's are real and the μ's complex. Since the coefficients of μ^2 and μ^0 in equation (15) are equal,

$$\mu_1\mu_2 = 1 \tag{19'}$$

always, and when the μ's are complex, $|\mu_1| = |\mu_2| = 1$. Thus, each wave travels through the stack with unchanging amplitude but with its phase being shifted by an equal amount for each layer that is passed. The accompanying wave in the opposite direction has a different amplitude ($|h| \neq 1$) so that there is a finite transfer of energy through the stack, and the reflexion coefficient of the whole stack is less than unity however many plates it contains.

When $\cos^2\phi < r^2$ and the μ's are real, the convenient expressions for μ and h (upper signs for μ_1 and h_1) are

$$\mu = \frac{\cos 2\phi - r^2 \pm 2\sqrt{[\sin^2\phi\,(r^2-\cos^2\phi)]}}{1-r^2}, \tag{20}$$

or

$$\mu = -\frac{\sin^2\phi}{1-r^2}\left(1 \mp \sqrt{\frac{[r^2-\cos^2\phi]}{\sin^2\phi}}\right)^2, \tag{21}$$

and

$$h = \frac{1}{r}\left(-\cos\phi \pm \frac{i}{\sin\phi}\sqrt{[\sin^2\phi\,(r^2-\cos^2\phi)]}\right). \tag{22}$$

With this convention for the signs, $|\mu_1| < 1$ irrespective of the signs of r and $\sin\phi$, so that the (μ_1, h_1) solution is always the one with amplitude decreasing in the downward direction through the stack.

When $\cos^2\phi > r^2$ and the μ's are complex, more convenient expressions are:

$$\mu = \frac{\cos 2\phi - r^2 \mp i\sin 2\phi\sqrt{[1-r^2/\cos^2\phi]}}{1-r^2}, \tag{23}$$

or

$$\mu = \exp(\mp i\theta) \tag{24}$$

where θ is real; this form is permissible because $|\mu| = 1$ as pointed out earlier. Equating real parts of equations (23) and (24) shows that θ is given by

$$\cos\theta = \frac{\cos 2\phi - r^2}{1-r^2}; \tag{25}$$

to give the correct sign to the imaginary part in equation (23), the solution to be chosen is the one which makes $\frac{1}{2}\theta$ lie in the same quadrant as ϕ.

The corresponding equation for h is

$$h = \frac{\cos\phi}{r}\left(-1 \pm \sqrt{[1-r^2/\cos^2\phi]}\right). \tag{26}$$

The upper sign makes $|h| < 1$ so that the (μ_1, h_1) solution is the one in which the net energy flux is downwards.

4.4. *Boundary conditions at bottom of the stack*

Equations (13) and (14) contain an arbitrary constant α. The appropriate value for this constant can be obtained by considering the boundary condition at the bottom of the stack. a_p^- represents the amplitude of the upward wave below the pth, or lowest, plate. In the case we are now considering there is no reflecting surface below this level, so this wave is non-existent and $a_p^- = 0$. Substituting this in equation (14) gives

$$0 = a_0^+ (\alpha h_1 \mu_1^p + (1-\alpha) h_2 \mu_2^p),$$

whence

$$\alpha = \frac{h_2 \mu_2^p}{h_2 \mu_2^p - h_1 \mu_1^p},$$

or

$$\alpha = \frac{h_2}{h_2 - m^2 h_1} \tag{27}$$

where

$$m = \mu_1^p = (1/\mu_2)^p. \tag{28}$$

4.5. *Amplitude reflexion coefficient of whole stack*

a_0^- represents the upward (reflected) wave above the first or topmost plate of the stack, while a_0^+ is the incident amplitude. Denoting the reflexion coefficient of the whole stack by R, we have, from equation (14):

$$R = \frac{a_0^-}{a_0^+} = \alpha h_1 + (1-\alpha) h_2.$$

Substituting for α from equation (27) gives

$$R = \frac{h_1 h_2 (1-m^2)}{h_2 - m^2 h_1}.$$

Now $h_1 h_2 = 1$ (equation (19)), so

$$R = \frac{1-m^2}{h_2 - m^2 h_1}. \tag{29}$$

It is convenient to rewrite this as

$$R = \frac{1-m^2}{\frac{1}{2}(h_1+h_2)(1-m^2) - \frac{1}{2}(h_1-h_2)(1+m^2)} \tag{30}$$

since $\frac{1}{2}(h_1+h_2)$ and $\frac{1}{2}(h_1-h_2)$ are simple expressions obtained from equations (12), (17), (22) or (26).

In the same way, the transmission coefficient of the whole stack is

$$T = \frac{a_p^+}{a_0^+} = \alpha \mu_1^p + (1-\alpha)\mu_2^p$$

$$= \frac{(h_2-h_1)m}{h_2 - m^2 h_1} = \frac{(h_2-h_1)m}{\frac{1}{2}(h_1+h_2)(1-m^2) - \frac{1}{2}(h_1-h_2)(1+m^2)}. \tag{31}$$

4.6. *Intensity reflexion coefficient of whole stack*

This quantity (reflectance, the fraction of the incident light intensity that is reflected) is the square of the modulus of R. It is convenient to use different procedures for obtaining the modulus when the μ's are real and when they are complex.

(*a*) *Case when μ's are real* ($\cos^2\phi < r^2$)

From equation (21) it follows that

$$m^2 = \left(\frac{\mu_1}{\mu_2}\right)^p = \left(\frac{1-\sqrt{\dfrac{r^2-\cos^2\phi}{\sin^2\phi}}}{1+\sqrt{\dfrac{r^2-\cos^2\phi}{\sin^2\phi}}}\right)^{2p}. \tag{32}$$

This quantity is real, and lies between 0 and 1.

(h_1+h_2) is real and (h_1-h_2) is imaginary, so from equation (30)

$$|R|^2 = \frac{(1-m^2)^2}{[\frac{1}{2}(h_1+h_2)]^2\,(1-m^2)^2-[\frac{1}{2}(h_1-h_2)]^2\,(1+m^2)^2}. \tag{33}$$

$[\frac{1}{2}(h_1+h_2)]^2 = 1+[\frac{1}{2}(h_1-h_2)]^2$ since $h_1h_2 = 1$ (equation (19)), so

$$|R|^2 = 1\Big/1+[\tfrac{1}{2}(h_1-h_2)]^2\left(1-\left(\frac{1+m^2}{1-m^2}\right)^2\right).$$

$[\frac{1}{2}(h_1-h_2)]^2 = k^2-1$ where $2k$ is the coefficient of h in equations (12) or (17), so

$$|R|^2 = 1\Big/1+\frac{4m^2\,(1-k^2)}{(1-m^2)^2}. \tag{34}$$

In equation (17), $k = \cos\phi/r$, so in this case

$$|R|^2 = 1\Big/1+\frac{4m^2\,(r^2-\cos^2\phi)}{r^2\,(1-m^2)^2}. \tag{35}$$

With an infinite number of plates m approaches zero and equation (35) shows that the reflectance then approaches unity, i.e. all the incident light is reflected.

(*b*) *Case where μ_1 and μ_2 are complex* ($\cos^2\phi > r^2$)

Using for μ the expression given in equation (24), we obtain

$$m^2 = \exp(-2ip\theta) \tag{36}$$

(θ being defined by equation (25)), and equation (30) becomes

$$R = \frac{1-\exp(-2ip\theta)}{\frac{1}{2}(h_1+h_2)\,(1-\exp(-2ip\theta))\;-\frac{1}{2}(h_1-h_2)\,(1+\exp(-2ip\theta))}$$

$$= \frac{1}{\frac{1}{2}(h_1+h_2)+i[\frac{1}{2}(h_1-h_2)]\cot p\theta}. \tag{37}$$

(h_1-h_2) is now real, so

$$|R|^2 = \frac{1}{[\frac{1}{2}(h_1+h_2)]^2+[\frac{1}{2}(h_1-h_2)]^2\cot^2 p\theta}.$$

As before, $[\frac{1}{2}(h_1+h_2)]^2 = [\frac{1}{2}(h_1-h_2)]^2+1$ since $h_1h_2 = 1$, and $[\frac{1}{2}(h_1-h_2)]^2 = k^2-1$, so

$$|R|^2 = \frac{1}{1+(k^2-1)\operatorname{cosec}^2 p\theta}. \tag{38}$$

From equation (17), $k = \cos\phi/r$, so in this case

$$|R|^2 = 1\Big/1+\frac{\cos^2\phi-r^2}{r^2\sin^2 p\theta}. \tag{39}$$

As the wavelength is changed, ϕ, and therefore also θ, changes, and this function fluctuates, the fluctuations becoming closer as the number of plates, or their optical thickness, is increased. This behaviour of multilayer dielectric reflectors is well known (e.g. Vašiček, 1960). The reflectance varies between zero when $\sin p\theta = 0$ and $r^2/\cos^2\phi$ when $\sin p\theta = \pm 1$.

For a thick stack of plates it is useful to calculate the mean reflectance $\overline{|R|^2}$ over one cycle of this fluctuation. Equation (39) may be rewritten.

$$|R|^2 = 1 - \frac{2(\cos^2\phi - r^2)}{2(\cos^2\phi - r^2) + r^2 - r^2\cos 2p\theta},$$

whence

$$\overline{|R|^2} = 1 - \frac{1}{2\pi}\int_{2p\bar{\theta}-\pi}^{2p\bar{\theta}+\pi} \frac{2(\cos^2\phi - r^2)\,dx}{2(\cos^2\phi - r^2) + r^2 - r^2\cos x}$$

$$= 1 - \sqrt{(1 - r^2/\cos^2\phi)}, \tag{40}$$

if ϕ can be considered as constant within the limits of integration. From equation (25) it can be seen that the range of $\pm\pi$ in $2p\theta$ corresponds to less than $\pm\dfrac{\pi}{4p}$ in ϕ.

(c) *Limiting case when* $\cos^2\phi = r^2$

This may be approached from either equation (35) or equation (39); the result is the same in both cases but the approach from equation (35) is the simpler.

Since $(r^2 - \cos^2\phi)$ is small, the right-hand side of equation (32) may be expanded by the binomial theorem, giving as the first two terms

$$m^2 = 1 - 4p\sqrt{\frac{r^2 - \cos^2\phi}{\sin^2\phi}}. \tag{41}$$

This can be taken as 1 except in the term $(1 - m^2)$, which becomes

$$4p\sqrt{\frac{r^2 - \cos^2\phi}{\sin^2\phi}}.$$

Substituting these results in equation (35) gives

$$|R|^2 = 1\Big/1 + \frac{\sin^2\phi}{4p^2r^2}.$$

Since $r^2 = \cos^2\phi$, this may be written as either

$$|R|^2 = 1\Big/1 + \frac{\tan^2\phi}{4p^2} \tag{42}$$

or

$$|R|^2 = 1\Big/1 + \frac{1 - r^2}{4p^2r^2}. \tag{43}$$

5. EXTENSION TO MORE GENERAL CASES

5.1. *Optical thicknesses of plates and spaces not equal* ($\phi_a \neq \phi_b$)

The appropriate expressions for ρ and τ have already been derived (equations (1*b*) and (2*b*)).

Equations (4)–(14) inclusive are valid whether or not $\phi_a = \phi_b$. The equations equivalent to equations (15), (16) and (17) can be obtained by substituting from equations (1*b*) and (2*b*) into equations (10), (11) and (12), obtaining

$$\mu^2 - 2\frac{\{\cos(\phi_a+\phi_b) - r^2\cos(\phi_a-\phi_b)\}}{1-r^2}\mu + 1 = 0, \tag{44}$$

$$h = \frac{-[\sin(\phi_a+\phi_b) - r^2\sin(\phi_a-\phi_b)] + i\{\mu(1-r^2) - [\cos(\phi_a+\phi_b) - r^2\cos(\phi_a-\phi_b)]\}}{2r\sin\phi_b} \tag{45}$$

and

$$h^2 + \frac{\{\sin(\phi_a+\phi_b) - r^2\sin(\phi_a-\phi_b)\}}{r\sin\phi_b}h + 1 = 0. \tag{46}$$

It is laborious to show directly from equations (44) and (46) that whenever μ is real, h is complex, and vice versa, as is the case when $\phi_a = \phi_b$, but this can be seen easily from equation (45). Writing $2k'$ for the coefficient of μ in equation (44),

$$\mu = \frac{\cos(\phi_a+\phi_b) - r^2\cos(\phi_a-\phi_b)}{1-r^2} \pm \sqrt{(k'^2-1)}.$$

Substituting this into equation (45) gives

$$h = \frac{-[\sin(\phi_a+\phi_b) - r^2\sin(\phi_a-\phi_b)] \pm i(1-r^2)\sqrt{(k'^2-1)}}{2r\sin\phi_b},$$

so that $k'^2 > 1$ is the condition both that μ is real and that h is complex.

This condition is satisfied by either $k' > +1$ or $k' < -1$. The former reduces to

$$\cos^2\tfrac{1}{2}(\phi_a+\phi_b) < r^2\cos^2\tfrac{1}{2}(\phi_a-\phi_b), \tag{47}$$

which becomes the same as equation (18) when $\phi_a = \phi_b$. The other range for real μ, $k' < -1$, reduces to

$$\sin^2\tfrac{1}{2}(\phi_a+\phi_b) < r^2\sin^2\tfrac{1}{2}(\phi_a-\phi_b), \tag{48}$$

which is now a finite range of $(\phi_a+\phi_b)$ centred around any even multiple of π. When $\phi_a = \phi_b$, $\sin\frac{1}{2}(\phi_a-\phi_b) = 0$ so this range is reduced to zero extent in the case treated in §§ 3 and 4.

Equations (19) and (19′) are still valid. The equivalents of equations (20)–(23) and (26) are too cumbersome to be useful. Equation (24) is still valid provided that the definition of θ (equation (25)) is modified to

$$\cos\theta = -k' = \frac{\cos(\phi_a+\phi_b) - r^2\cos(\phi_a-\phi_b)}{1-r^2}; \tag{49}$$

the solution to be chosen is the one which makes $\frac{1}{2}\theta$ lie in the same quadrant as $\frac{1}{2}(\phi_a+\phi_b)$.

Equations (27)–(31), (33), (34), (36)–(38) are still valid provided that $2k$ is taken as the coefficient of h in equation (46) instead of in equation (17), and θ is defined by equation (49) instead of equation (25). The signs to be chosen in evaluating the μ's and h's should be selected by the following criteria:

(a) When the μ's are real, μ_1 is the solution of equation (44) whose absolute value is less than unity. The correct sign for h_1 is obtained by substituting μ_1 into equation (45).

(b) When the μ's are complex, h_1 is the solution of equation (46) whose absolute value is less than unity. The correct sign for μ_1 is obtained by substituting h_1 into equation (45) and rearranging to obtain μ.

Calculation of the reflectance of the whole stack is a little more laborious than when $\phi_a = \phi_b$, but it can be carried out by means of equations (34) or (38). When μ is real ($|k'| > 1$), equation (34) is used. k', the coefficient of 2μ in equation (44), must be evaluated, and from it, μ_1 and μ_2. m^2 is then obtained as $(\mu_1/\mu_2)^p$. k, the coefficient of $2h$ in equation (46), is evaluated, and k and m^2 are substituted into equation (34), giving the required result. When μ is complex, θ is evaluated from equation (49), and k and θ are substituted into equation (38).

5.2. *Oblique incidence*

When the incident light is not directed along the normal to the reflecting surfaces in the stack, different expressions must be used for r, ϕ_a and ϕ_b. Equations (1b) and (2b) will then give the new values for ρ and τ, and the rest of the treatment is unchanged.

Let the angle between the normal and the direction of propagation be ν_a in the spaces (refractive index n_a) and ν_b in the plates (index n_b). r differs according to the direction of polarization of the incident light, and is given by Fresnel's equations:

$$r_{\parallel} = \frac{-\tan(\nu_a - \nu_b)}{\tan(\nu_a + \nu_b)} \tag{50}$$

when the electric vector is in the plane of incidence, and

$$r_{\perp} = \frac{+\sin(\nu_a - \nu_b)}{\sin(\nu_a + \nu_b)} \tag{51}$$

when it is perpendicular to the plane of incidence (see, for example, Born & Wolf (1964) p. 40, and footnote on p. 41 for the difference between the signs of $r_{\parallel}$ and $r_{\perp}$). The phase retardations for a single passage through each plate and space are now

$$\phi_b = \frac{2\pi}{\lambda} n_b d_b \cos \nu_b \tag{52}$$

and

$$\phi_a = \frac{2\pi}{\lambda} n_a d_a \cos \nu_a \tag{53}$$

respectively (cf. Born & Wolf, 1964, p. 282).

Note that if $\phi_a = \phi_b$ for normal incidence, this will no longer be true for obliquely incident light.

5.3. *Additional reflecting surfaces present*

The situation to be considered is shown in Fig. 4, where the stack is deposited on a material of refractive index n_s and may also be covered with a material of refractive index n_0. r_s and r_0 are the amplitude reflexion coefficients for n_a/n_s and n_0/n_a interfaces respectively.

The whole stack, from the upper to the lower broken line in Fig. 4, can be regarded as a single partially reflecting surface with reflexion and transmission coefficients R and T defined by equations (30) and (31).

The combined reflexion coefficient R_s due to the stack and the n_a/n_s interface at the bottom can be obtained by means of equation (1). Since the stack (including the half-spaces at both ends) is symmetrical, R and T are the same for light incident in either direction, i.e. $R' = R$ and $T' = T$. Hence, putting $r_1 = r_1' = R$ and $r_2 = r_s$ in equation (1),

$$R_s = \frac{R + r_s (T^2 - R^2) \exp(-2i\phi_s)}{1 - Rr_s \exp(-2i\phi_s)}, \tag{54}$$

where $$\phi_s = \frac{2\pi}{\lambda}\left(d_p - \frac{d_a}{2}\right) n_a \cos \nu_a$$

is the extra phase lag due to material of refractive index n_a below the lower broken line.

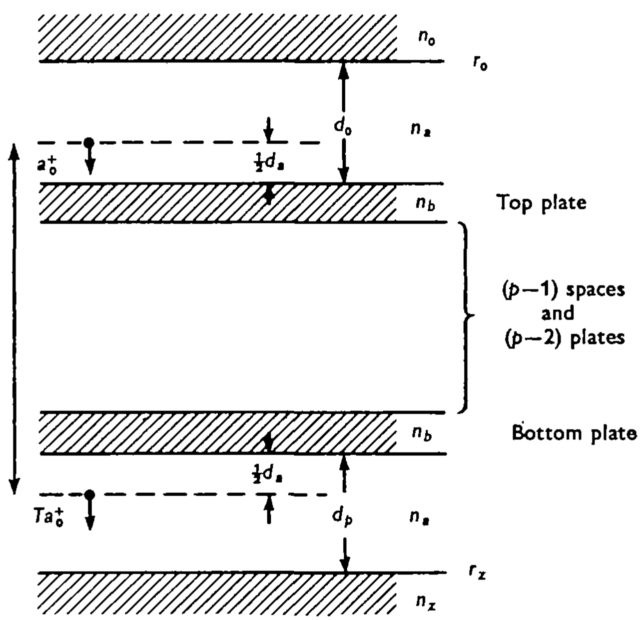

Fig. 4. Diagram illustrating the situations dealt with in § 5.3. The arrow on the left indicates the extent of the stack for which R and T are the reflexion and transmission coefficients.

It can be seen from equations (30) and (31) that R and T are always $\frac{1}{2}\pi$ different in phase; if $T = |T| \exp(-i\chi)$ then $R = \pm i|R| \exp(-i\chi)$ and $T^2 - R^2 = \exp(-2i\chi)$ since $|T|^2 + |R|^2 = 1$ for the conservation of energy. Hence, equation (54) becomes

$$R_s = \frac{R + r_z \exp(-2i(\phi_s + \chi))}{1 - Rr_s \exp(-2i\phi_s)}. \tag{55}$$

If the n_0/n_a interface at the top is also present, then equation (1) can be applied again; this time $r_1 = r_0$, $r_1' = -r_0$ and $t_1t_1' - r_1r_1' = 1$, while r_2 is R_s. Hence, the combined reflexion coefficient R_{0s} of the whole system is

$$R_{0s} = \frac{r_0 + R_s \exp(-2i\phi_0)}{1 + r_0 R_s \exp(-2i\phi_0)}, \tag{56}$$

where

$$\phi_0 = \frac{2\pi}{\lambda}\left(d_0 - \frac{d_a}{2}\right) n_a \cos \nu_a.$$

If these results are used when the incident light is oblique, it is important that the same sign convention should be used throughout for r_1 (see equation (50)).

R_s and R_{0s}, like R, are complex quantities indicating the phase as well as the absolute amplitude of the reflected light. To obtain the corresponding reflectances, it is necessary to evaluate R_s or R_{0s} as a complex quantity and take the sum of the squares of its real and imaginary parts.

5.4. *More complex sequences*

It has so far been assumed that each repeat in the stack consists of one plate of refractive index n_b and one space of refractive index n_a. If the repeating unit is more complex, the reflexion and transmission coefficients ρ and τ for a single unit can be obtained by repeated application of equations (1) and (2); this is the procedure described for example by Vašiček (1960), and used in § 5.3. μ, h, k and k' are then found by substituting this ρ and τ into equations (10), (11) and (12), and from them, m and θ. The reflectance of the stack is then found from equation (34) or (38) according as the μ's are real or complex.

SUMMARY

1. A convenient method is presented for calculating the reflectance of a stack of dielectric layers consisting of a series of identical repeats of any particular sequence of layers. The method is closely related to that published by Lord Rayleigh in 1917.

2. In this method, two quadratic equations are formed from the thicknesses and refractive indices of the layers composing a single repeat unit. The reflectance is obtained by substituting the solutions of these equations into an explicit formula.

3. Particularly simple formulae result for the case of a stack of p plates, optical thickness $\lambda\phi/2\pi$, uniformly spaced in an infinite medium with spaces of the same optical thickness. If r is the amplitude reflexion coefficient at a single interface, the reflectance of the whole stack is as follows:

(*a*) when $\cos^2\phi < r^2$,

$$\text{reflectance} = 1 \bigg/ 1 + \frac{4m^2(r^2 - \cos^2\phi)}{r^2(1-m^2)^2},$$

where

$$m = \left(\frac{1 - \sqrt{\dfrac{r^2 - \cos^2\phi}{\sin^2\phi}}}{1 + \sqrt{\dfrac{r^2 - \cos^2\phi}{\sin^2\phi}}}\right)^p;$$

(*b*) when $\cos^2\phi > r^2$,

$$\text{reflectance} = 1 \bigg/ 1 + \frac{\cos^2\phi - r^2}{r^2 \sin^2 p\theta},$$

where $$\cos\theta = \frac{\cos 2\phi - r^2}{1 - r^2};$$

(*c*) when the number of repeats in the stack is large ($p \to \infty$), reflexion is complete so long as $\cos^2\phi < r^2$. Outside this range the reflectance is $1 - \sqrt{(1 - r^2/\cos^2\phi)}$.

4. These results are extended to cover: (*a*) unequal optical thicknesses in plates and spaces; (*b*) oblique incidence; (*c*) layers of materials of other refractive indices above and below the stack itself; and (*d*) stacks consisting of repeats of more complex units.

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